

A reinforcement of the Bourgain-Kontorovich's theorem-II.

D. A. Frolenkov* I. D. Kan,[†]

Abstract

Zaremba's conjecture (1971) states that every positive integer number d can be represented as a denominator (continuant) of a finite continued fraction $\frac{b}{d} = [d_1, d_2, \dots, d_k]$, whose partial quotients d_1, d_2, \dots, d_k belong to a finite alphabet $\mathcal{A} \subseteq \mathbb{N}$. In this paper it is proved for an alphabet \mathcal{A} , such that the Hausdorff dimension $\delta_{\mathcal{A}}$ of the set of infinite continued fractions whose partial quotients belong to \mathcal{A} , that the set of numbers d , satisfying Zaremba's conjecture with the alphabet \mathcal{A} , has positive proportion in \mathbb{N} . The result improves our previous reinforcement of the corresponding Bourgain-Kontorovich's theorem.

Bibliography: 8 titles.

Keywords: continued fraction, continuant, the circle method, exponential sums.

1 Introduction

Let \mathfrak{R}_A be the set of rational numbers whose continued fraction expansion has all partial quotients being bounded by A :

$$\mathfrak{R}_A = \left\{ \frac{b}{d} = [d_1, d_2, \dots, d_k] \mid 1 \leq d_j \leq A \text{ для } j = 1, \dots, k \right\}.$$

Let \mathfrak{D}_A be the set of denominators of numbers in \mathfrak{R}_A : $\mathfrak{D}_A = \left\{ d \mid \exists b : (b, d) = 1, \frac{b}{d} \in \mathfrak{R}_A \right\}$. And set

$$\mathfrak{D}_A(N) = \left\{ d \in \mathfrak{D}_A \mid d \leq N \right\}.$$

Conjecture 1.1. (*Zaremba's conjecture* [4, p. 76], 1971). For sufficiently large A one has

$$\mathfrak{D}_A = \mathbb{N}.$$

Bourgain and Kontorovich suggested that the problem should be generalized in the following way. Let $\mathcal{A} \in \mathbb{N}$ be any finite alphabet ($|\mathcal{A}| \geq 2$) and let $\mathfrak{R}_{\mathcal{A}}$ and $\mathfrak{C}_{\mathcal{A}}$ be the set of finite and infinite continued fractions whose partial quotients belong to \mathcal{A} . And let $\mathfrak{D}_{\mathcal{A}}(N) = \left\{ d \mid d \leq N, \exists b : (b, d) = 1, \frac{b}{d} \in \mathfrak{R}_{\mathcal{A}} \right\}$ be the set of denominators bounded by N . Let $\delta_{\mathcal{A}}$ be the Hausdorff dimension of the set $\mathfrak{C}_{\mathcal{A}}$. In the paper [5] we proved the following theorem on the basis of the method devised by Bourgain-Kontorovich [1].

*Research is supported by RFFI (grant № 12-01-31165)

[†]Research is supported by RFFI (grant № 12-01-00681-a)

Theorem 1.1. *For any alphabet \mathcal{A} with*

$$\delta_{\mathcal{A}} > 1 - \frac{5}{\sqrt{369} + 23} = 0,8815\dots, \quad (1.1)$$

the following inequality (positive proportion) holds

$$\#\mathfrak{D}_{\mathcal{A}}(N) \gg N \quad (1.2)$$

The main result of the paper is the following theorem.

Theorem 1.2. *For any alphabet \mathcal{A} with*

$$\delta_{\mathcal{A}} > \frac{7}{8} = 0,875, \quad (1.3)$$

the inequality (1.2) holds.

The paper is a sequel of our article [5]. So we will heavily refer to statements and constructions in [5]. It should be mentioned that the proof of the Theorem 1.2 repeats significantly the proof of the Theorem 1.1 in [5].

Throughout $\epsilon_0 = \epsilon_0(\mathcal{A}) \in (0, \frac{1}{2500})$. For two functions $f(x), g(x)$ the Vinogradov notation $f(x) \ll g(x)$ means that there exists a constant C , depending on A , such that $|f(x)| \leq Cg(x)$. Also a traditional notation $e(x) = \exp(2\pi ix)$ is used. The cardinality of a finite set S is denoted either $|S|$ or $\#S$. $[\alpha]$ and $\|\alpha\|$ denote the integral part of α and the distance from α to the nearest integer respectively.

2 Estimates of exponential sums.

We define the exponential sum $S_N(\theta)$ as follows

$$S_N(\theta) = \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|), \quad (2.1)$$

where $\Omega_N = \Omega_N(\mathcal{A})$ is a proper set of matrices (ensemble) constructed in [5, глава II]. We use the following norm $\|\gamma\| = \max\{|a|, |b|, |c|, |d|\}$ for the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. One can find some more facts about this norm in [5, §5]. It was obtained in [5, §7] that to prove the inequality (1.2) it is sufficient to obtain the following estimate

$$\int_0^1 |S_N(\theta)|^2 d\theta \ll \frac{1}{N} |\Omega_N|^2. \quad (2.2)$$

It follows from the Dirichlet's theorem that for any $\theta \in [0, 1]$ there exist $a, q \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$ such that

$$\theta = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq N^{1/2}, \quad \beta = \frac{K}{N}, \quad |K| \leq \frac{N^{1/2}}{q}, \quad (2.3)$$

with $a = 0$ and $a = q$ being possible if only $q = 1$. The purpose of the following reasonings is a slight modification of the results in [5, §12].

Lemma 2.1. ([5, Lemma 16.1.]) *The following inequality holds*

$$\int_0^1 |S_N(\theta)|^2 d\theta \leq \frac{1}{N} \sum_{0 \leq a \leq q \leq N^{1/2}}^* \int_{|K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK, \quad (2.4)$$

where \sum^* means that the sum is taken over a and q being coprime for $q \geq 1$, and $a = 0, 1$ for $q = 1$.

It follows from the statement of Lemma 2.1 that we need to know how to estimate the following expression

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \int_{|K| \leq Y} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK, \quad (2.5)$$

where Y may depend on q . The following reasonings are similar to [8, Lemma 26 p.145]. Let take a sufficiently large number T , then

$$\int_{|K| \leq Y} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \leq \sum_{|l| \leq TY} \int_{l/T}^{(l+1)/T} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK. \quad (2.6)$$

Hence, in any interval $[l/T, (l+1)/T]$ we have

$$K = \frac{l}{T} + \lambda, \quad 0 \leq \lambda \leq \frac{1}{T}, \quad \theta = \frac{a}{q} + \frac{K}{N} = \frac{a}{q} + \frac{l}{TN} + \frac{\lambda}{N},$$

$$\left| S_N(\theta) - S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right| \ll \lambda |\Omega_N| \Rightarrow |S_N(\theta)|^2 \ll \left| S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right|^2 + \lambda^2 |\Omega_N|^2. \quad (2.7)$$

So

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \int_{|K| \leq Y} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \leq \frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \sum_{|l| \leq TY} \left(\frac{1}{T} \left| S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right|^2 + \frac{|\Omega_N|^2}{T^3} \right) \quad (2.8)$$

Choosing T sufficiently large we obtain that the investigation of the expression of the form (2.5) reduce to the investigation of the quantity

$$\frac{1}{TN} \sum_{0 \leq a \leq q \leq X}^* \sum_{|l| \leq TY} \left| S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right|^2. \quad (2.9)$$

Our next purpose is to modify Lemma 12.4. [5]. We formulate the following theorem for convenience of the reader. Let

$$Q_0 = \max \left\{ \exp \left(\frac{10^5 A^4}{\epsilon_0^2} \right), \exp(\epsilon_0^{-5}) \right\}. \quad (2.10)$$

Theorem 2.1. [Theorem 11.5 [5]] For any $M^{(1)}$ and $M^{(3)}$ such that

$$Q_0 \leq M^{(1)}, M^{(3)} \leq \frac{N}{Q_0}, M^{(1)}M^{(3)} < N^{1-\epsilon_0}, \quad (2.11)$$

the ensemble Ω_N can be represented in the form $\Omega_N = \Omega^{(1)}\Omega^{(2)}\Omega^{(3)}$, such that for any matrices $\gamma_1 \in \Omega^{(1)}$, $\gamma_2 \in \Omega^{(2)}$, $\gamma_3 \in \Omega^{(3)}$ the following inequalities hold

$$M^{(1)} \ll \|\gamma_1\| \ll (M^{(1)})^{1+2\epsilon_0}, M^{(3)} \ll \|\gamma_3\| \ll (M^{(3)})^{1+2\epsilon_0}, \quad (2.12)$$

$$\frac{N}{(M^{(1)}M^{(3)})^{1+2\epsilon_0}} \ll \|\gamma_2\| \ll \frac{N}{M^{(1)}M^{(3)}}. \quad (2.13)$$

We denote $\hat{l} = \max\{1, \frac{|l|}{T}\}$ and

$$P_{Q_1, Q}^{\kappa_1, \kappa} = \left\{ \theta = \frac{a}{q} + \frac{l}{TN} \mid (a, q) = 1, 0 \leq a \leq q, Q_1 \leq q \leq Q, \kappa_1 \leq l \leq \kappa \right\}. \quad (2.14)$$

For an arbitrary subset $Z \subseteq P_{Q_1, Q}^{\kappa_1, \kappa}$, we denote

$$\mathfrak{M}^*(g_2) = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (2.16) \text{ и } (2.17) \text{ выполнены} \right\}, \quad (2.15)$$

where $\theta^{(i)} = \frac{a^{(i)}}{q^{(i)}} + \frac{l^{(i)}}{TN}$, $i = 1, 2$ and

$$|g_2 g_3^{(1)} \frac{l^{(1)}}{TN} - g_2 g_3^{(2)} \frac{l^{(2)}}{TN}|_{1,2} \leq \frac{1}{M^{(1)}}, \quad (2.16)$$

$$\|g_2 g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_2 g_3^{(2)} \frac{a^{(2)}}{q^{(2)}}\|_{1,2} = 0. \quad (2.17)$$

We recall that the subscripts "1,2" mean that the property holds for both coordinates. The following lemma can be proved in the same manner as Lemma 12.4. in [5].

Lemma 2.2. Let $Z \subseteq P_{Q_1, Q}^{\kappa_1, \kappa}$. Let $M^{(1)}$ and $M^{(3)}$ satisfy the condition of the 2.1, and let the inequality

$$M^{(1)} > 150A^2[q^{(1)}, q^{(2)}] \max_{\kappa_1 \leq l \leq \kappa} \hat{l} \quad (2.18)$$

holds for any $\theta^{(1)}, \theta^{(2)} \in Z$. Then the following bound holds

$$\sum_{\theta \in Z} |S_N(\theta)| \ll (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} \sum_{g_2 \in \tilde{\Omega}^{(2)}} |\mathfrak{M}^*(g_2)|^{1/2}. \quad (2.19)$$

Let state one more lemma of a general nature that was proved in [5, §12]. A similar statement was used by S.V. Konyagin in [7, 17].

Lemma 2.3. Let W be a finite subset of the interval $[0, 1]$ and let $|W| > 10$. Let $f : W \rightarrow \mathbb{R}_+$ be a function such that, for any subset $Z \subseteq W$ the following bound holds

$$\sum_{\theta \in Z} f(\theta) \leq C_1 |Z|^{1/2} + C_2,$$

where C_1, C_2 are non-negative constants not depending on the set Z . Then the following estimate holds

$$\sum_{\theta \in W} f^2(\theta) \ll C_1^2 \log |W| + C_2 \max_{\theta \in W} f(\theta) \quad (2.20)$$

with the absolute constant in Vinogradov symbol.

3 «The case $\mu = 3$.»

This section corresponds to the section 14 in [5]. So it has the same title. We formulate some results of [5, §14] required for proving the estimate (2.2). We also prove a number of lemmas reinforcing the results of [5, §14]. The following lemma is a modification of Lemma 14.1. and 14.2. in [5]. We write $\overline{K} = \max\{1, |K|\}$.

Lemma 3.1. *If $\frac{\kappa}{T}Q^{2,5} \leq N^{1-\epsilon_0}$, $\frac{\kappa_1}{T} \geq Q_0$, then the following bound holds*

$$\frac{1}{T} \sum_{\theta \in P_{Q_1, Q}^{\kappa_1, \kappa}} |S_N(\theta)|^2 \ll |\Omega_N|^2 \left(Q^2 \frac{\kappa}{T}\right)^{2\gamma+7\epsilon_0} Q_1^{\gamma-1+5\epsilon_0}. \quad (3.1)$$

Proof. Let $Z \subseteq P_{Q_1, Q}^{\kappa_1, \kappa}$ be any subset. In the same way as in Lemma 14.1. of [5] we obtain that to satisfy the conditions (2.16) и (2.17) it is necessary to have $q^{(1)} = q^{(2)} = \mathbf{q}$. Then the conditions (2.16) and (2.17) can be written as

$$g_3^{(1)} a^{(1)} \equiv g_3^{(2)} a^{(2)} \pmod{\mathbf{q}}, \quad |g_2(g_3^{(1)} \frac{l^{(1)}}{TN} - g_3^{(2)} \frac{l^{(2)}}{TN})|_{1,2} \leq \frac{1}{M^{(1)}}. \quad (3.2)$$

We fix $\theta^{(1)}$ (that is $a^{(1)}, \mathbf{q}, l^{(1)}$ are fixed) for which there are $|Z|$ choices. After this we estimate the number of solutions of (3.2) independently of $\theta^{(1)}$. Then (see Lemma 14.8. in [5]) $a^{(2)}$ is uniquely determined and $x_1 y_2 \equiv x_2 y_1 \pmod{\mathbf{q}}$, where

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t.$$

In view of the Theorem 2.1, we obtain from (3.2) that

$$\#l^{(2)} \ll \frac{TN}{M^{(1)} \|g_2 g_3^{(2)}\|} \ll T(M^{(1)})^{2\epsilon_0}.$$

Hence,

$$|\mathfrak{M}^*(g_2)| \ll |Z| T(M^{(1)})^{2\epsilon_0} \sum_{g_3^{(1)}, g_3^{(2)} \in \Omega^{(3)}} \mathbf{1}_{\{x_1 y_2 \equiv x_2 y_1 \pmod{\mathbf{q}}\}} \quad (3.3)$$

We put

$$M^{(1)} = 150A^2 Q^2 \frac{\kappa}{T}, \quad M^{(3)} = \frac{Q_1^{1/2-2\epsilon_0}}{80A^3}. \quad (3.4)$$

Then the congruences in (3.3) turn into equations and we obtain $|\mathfrak{M}^*(g_2)| \ll |Z| T(M^{(1)})^{2\epsilon_0} |\Omega^{(3)}|$. Applying Lemma 2.2, we have

$$\sum_{\theta \in Z} |S_N(\theta)| \ll (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} \sum_{g_2 \in \tilde{\Omega}^{(2)}} |\mathfrak{M}^*(g_2)|^{1/2} \ll |\Omega^{(1)}|^{1/2} |\Omega^{(2)}| |\Omega^{(3)}|^{1/2} |Z|^{1/2} T^{1/2} (M^{(1)})^{1+3\epsilon_0}. \quad (3.5)$$

Hence, using the bound $|\Omega^{(i)}| \geq (M^{(i)})^{2\delta-\epsilon_0}$, proved in [5, (11.63)], we obtain

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |\Omega_N| \frac{(M^{(1)})^{1+3\epsilon_0}}{(M^{(1)} M^{(3)})^{\delta-\epsilon_0/2}} |Z|^{1/2} T^{1/2}.$$

Applying Lemma 2.3, we have

$$\sum_{\theta \in P_{Q_1, Q}^{\kappa_1, \kappa}} |S_N(\theta)|^2 \ll |\Omega_N|^2 T \frac{(M^{(1)})^{2\gamma+7\epsilon_0} (M^{(3)})^{2\gamma+\epsilon_0}}{(M^{(3)})^2}.$$

Using (3.4), we obtain (3.1). Lemma is proved. \square

We denote

$$P_{Q_1, Q}^{(\beta)} = \left\{ \theta = \frac{a}{q} + \beta \mid (a, q) = 1, 0 \leq a \leq q, Q_1 \leq q \leq Q \right\}. \quad (3.6)$$

Lemma 3.2. (see [5, Lemma 14.5.]) *Let the following inequalities hold*

$$N^{\epsilon_0/2} \leq Q^{1/2} \leq Q_1 \leq Q, \bar{K}Q \leq N^\alpha,$$

and $\alpha \leq \frac{1}{2} + \epsilon_0$. Then for any $Z \subseteq P_{Q_1, Q}^{(\beta)}$ the following bound holds

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |\Omega_N| |Z|^{1/2} \left(\frac{N^{1-\delta+3\epsilon_0}}{(\bar{K}Q_1)^{1/2}} + \frac{N^{1-\frac{3-\alpha}{2}\delta+4,5\epsilon_0}}{\bar{K}^{1/2}} \right) + |\Omega_N| N^{\frac{1+\alpha}{2}-\delta+2,5\epsilon_0} Q. \quad (3.7)$$

Proof. The inequality (3.7) is proved in the same manner as [5, (14.28)] with the use of Lemma 3.3, which will be proved below, instead of Lemma 14.8 in [5]. This completes the proof of the lemma. \square

We denote

$$\mathfrak{N}(g_2) = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (3.9) \text{ and } (3.10) \text{ hold} \right\}, \quad (3.8)$$

where

$$\|g_2 g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_2 g_3^{(2)} \frac{a^{(2)}}{q^{(2)}}\|_{1,2} \leq \frac{74A^2 \bar{K}}{M^{(1)}}, \quad (3.9)$$

$$|g_2 g_3^{(1)} - g_2 g_3^{(2)}|_{1,2} \leq \min \left\{ \frac{73A^2 N}{M^{(1)}}, \frac{73A^2 N}{M^{(1)} \bar{K}} + \frac{N}{\bar{K}} \left\| g_2 g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_2 g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \right\}. \quad (3.10)$$

It was proved in [5, лемма 12.3.] that

$$\sum_{\theta \in Z} |S_N(\theta)| \ll (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} \sum_{g_2 \in \Omega^{(2)}} |\mathfrak{N}(g_2)|^{1/2}. \quad (3.11)$$

Let $g_3^{(1)} = (x_1, x_2)^t$, $g_3^{(2)} = (y_1, y_2)^t$, $\mathcal{Y} = x_1 y_2 - y_1 x_2$. We represent the set $\mathfrak{N}(g_2)$ as the union of the sets $\mathfrak{M}_1, \mathfrak{M}_2$. For the first one we have $\mathcal{Y} = 0$, for the second one $\mathcal{Y} \neq 0$. It was proved in [5, лемма 14.7.] that $|\mathfrak{M}_1| \ll_\epsilon Q^{2+\epsilon} |\Omega^{(3)}|$. The following lemma is a modification of Lemma 14.8 in [5].

Lemma 3.3. (см. [5, Lemma 14.8.]) *Under the hypotheses of Lemma 3.2 one has*

$$|\mathfrak{M}_2| \ll |Z| \left(\frac{73A^2 N}{M^{(1)}} \right)^2 \frac{1}{\bar{K}Q_1} (|\Omega^{(3)}| N^{2\epsilon_0} + Q^{1+\epsilon_0}). \quad (3.12)$$

Proof. To simplify we denote $\mathcal{T} = \frac{73A^2N}{M^{(1)}}$. It was proved in [5, лемма 14.8.] that

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{\substack{g_3^{(2)} \in \Omega^{(3)} \\ |g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{\mathcal{T}}{K}}} \mathbf{1}_{\{x_1 y_2 \equiv x_2 y_1 \pmod{\mathbf{q}}\}}. \quad (3.13)$$

Changing the variables $z_1 = x_1 - y_1$, $z_2 = x_2 - y_2$, we obtain

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_{1,2}| \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathbf{q}}\}}. \quad (3.14)$$

We consider three cases.

1. Let $z_1 > 0$, $z_2 > 0$. We fix the vector $g_3^{(1)} \in \Omega^{(3)}$, then $x_1 z_2 - x_2 z_1 = j\mathbf{q}$. Let estimate the amount of j . We have

$$x_1 - x_2 \frac{\mathcal{T}}{K} \leq j\mathbf{q} \leq x_1 \frac{\mathcal{T}}{K} - x_2$$

and, hence, $\#j \ll \frac{\mathcal{T}^2}{\mathbf{q}K} + 1$. For a fixed j the solution of the congruence is given by the formulae

$$z_1 = z_{1,0} + nx_1, \quad z_2 = z_{2,0} + nx_2.$$

In view of $x_2 \gg \frac{\mathcal{T}}{(M^{(1)})^{2\epsilon_0}}$, we have $\#n \ll \frac{(M^{(1)})^{2\epsilon_0}}{K} + 1$. Thus,

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < z_{1,2} \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathbf{q}}\}} \ll |\Omega^{(3)}| \left(\frac{\mathcal{T}^2}{\mathbf{q}K} + 1 \right) \left(\frac{(M^{(1)})^{2\epsilon_0}}{K} + 1 \right). \quad (3.15)$$

It follows from the conditions of Lemma 3.2 that $\mathcal{T}^2 > \mathbf{q}K$, so one has

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < z_{1,2} \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathbf{q}}\}} \ll |\Omega^{(3)}| \frac{\mathcal{T}^2}{\mathbf{q}K} N^{2\epsilon_0}. \quad (3.16)$$

2. Let $z_1 > 0$, $z_2 < 0$. In the same way as in the previous case we obtain

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < -z_2, z_1 \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathbf{q}}\}} \ll |\Omega^{(3)}| \frac{\mathcal{T}^2}{\mathbf{q}K} N^{2\epsilon_0}. \quad (3.17)$$

3. Let $z_1 = 0$. One has

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_2| \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv 0 \pmod{\mathbf{q}}\}} \leq \sum_{g_3^{(1)} \in \Omega^{(3)}} \left(\frac{\mathcal{T}}{\mathbf{q}K} (x_1, \mathbf{q}) + 1 \right) \leq \quad (3.18)$$

$$\leq |\Omega^{(3)}| + \frac{\mathcal{T}^2}{\mathbf{q}K} \sum_{x_1 \leq \mathcal{T}} (x_1, \mathbf{q}). \quad (3.19)$$

Next

$$\sum_{x_1 \leq \mathcal{T}} (x_1, q) \leq \sum_{d|q} d \left(\frac{\mathcal{T}}{d} + 1 \right) \ll q^\epsilon \mathcal{T} + q^{1+\epsilon}$$

and so

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_2| \leq \frac{\mathcal{T}}{K}} \mathbf{1}_{\{x_1 z_2 \equiv 0 \pmod{\mathbf{q}}\}} \ll |\Omega^{(3)}| + \frac{\mathcal{T}^2}{qK} (q^\epsilon \mathcal{T} + q^{1+\epsilon}). \quad (3.20)$$

Using (3.16), (3.17) and (3.20) we obtain

$$|\mathfrak{M}_2| \ll |Z| \frac{\mathcal{T}^2}{qK} (q^\epsilon \mathcal{T} + q^{1+\epsilon} + |\Omega^{(3)}| N^{2\epsilon_0}) \leq |Z| \frac{\mathcal{T}^2}{qK} (q^{1+\epsilon} + |\Omega^{(3)}| N^{2\epsilon_0}). \quad (3.21)$$

Lemma is proved. \square

Based on Lemma 3.2 and using Lemma 2.3 we obtain in the same way as [5, следствию 14.1.] the following statement

Corollary 3.1. *Under the hypotheses of Lemma 3.2 one has*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 (C_1^2 Q^{\epsilon_0} + C_2'), \quad (3.22)$$

where

$$C_1 = \frac{N^{1-\delta+3\epsilon_0}}{(\overline{K}Q_1)^{1/2}} + \frac{N^{1-\frac{3-\alpha}{2}\delta+4,5\epsilon_0}}{\overline{K}^{1/2}}, \quad C_2' = \frac{N^{\frac{3+\alpha}{2}-2\delta+3,5\epsilon_0}Q}{(\overline{K}Q_1)^{1-2\epsilon_0}}. \quad (3.23)$$

4 Estimates for integrals of $|S_N(\theta)|^2$.

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1 in [5]. We will need a number of Lemmas from [5, §16] which will be presented without proof.

Lemma 4.1. (*[5, Lemma 16.2.]*) *The following inequality holds*

$$\begin{aligned} \int_0^1 |S_N(\theta)|^2 d\theta &\leq 2Q_0^2 \frac{|\Omega_N|^2}{N} + \frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK + \\ &\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq Q_0 \\ \frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}}}^* \int \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK + \frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{|K| \leq \frac{Q_0}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \end{aligned} \quad (4.1)$$

It is convenient to use the following notation

$$\gamma = 1 - \delta, \quad \xi_1 = N^{2\gamma+7\epsilon_0}. \quad (4.2)$$

The second and the third integral in the right side of (4.1) are estimated in the following lemma.

Lemma 4.2. (*[5, Lemmas 16.3, 16.4. and 16.5]*) For $\gamma < \frac{5}{36} - 6\epsilon_0$ and $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequalities hold

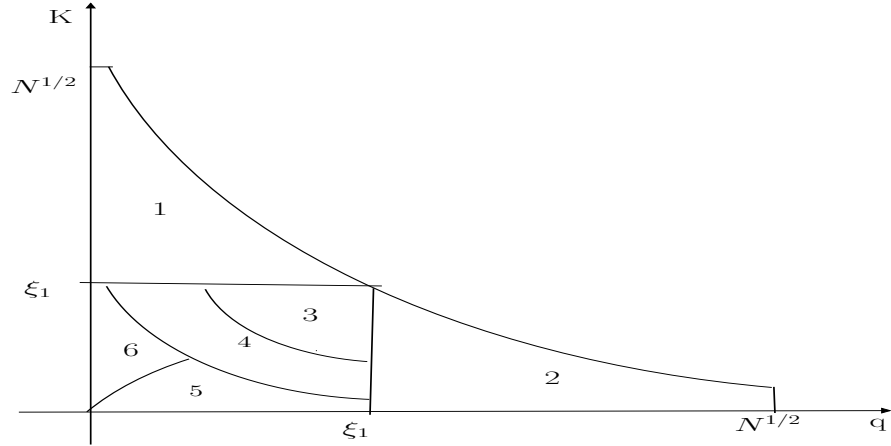
$$\frac{1}{N} \sum_{1 \leq a \leq q \leq N^{1/2}}^* \int_{|K| \leq \frac{Q_0}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.3)$$

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.4)$$

It remains to estimate the first integral in the right side of (4.1), that is,

$$\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \quad (4.5)$$

The following lemmas will be devoted to this. We partition the range of summation and integration over q, K into six subareas:



Lemma 4.3 corresponds to the domain 1, Lemma 4.5 corresponds to the domain 2, Lemma 4.6 corresponds to the domain 3, Lemma 4.7 corresponds to the domain 4, Lemma 4.9 corresponds to the domain 5, Lemma 4.8 corresponds to the domain 6.

To prove some lemmas we need the following parameters. Let $N \geq N_{\min} = N_{\min}(\epsilon_0, \mathcal{A})$, we denote

$$J = J(N) = \left\lceil \frac{\log \log N - 4 \log(10A) + 2 \log \epsilon_0}{-\log(1 - \epsilon_0)} \right\rceil, \quad (4.6)$$

where, as usual, $A \geq |\mathcal{A}| \geq 2$, and require the following inequality $J(N_{\min}) \geq 10$ to hold. Now let define a finite sequence $\{N_j\}$, having set $N_{J+1} = N$ and

$$N_j = \begin{cases} N^{\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{1-j}}, & \text{if } -1 - J \leq j \leq 1; \\ N^{1-\frac{1}{2-\epsilon_0}(1-\epsilon_0)^j}, & \text{if } 0 \leq j \leq J. \end{cases} \quad (4.7)$$

It is obvious that the sequence is well-defined for $j = 0$ and $j = 1$. A detailed description of properties of the sequence is given in [5, §9].

Lemma 4.3. (*[5, Lemma 16.6.]*) *The following inequality holds*

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\xi_1 \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.8)$$

Let

$$c_1 = c_1(N), c_2 = c_2(N), Q_0 \leq c_1 < c_2 \leq N^{1/2},$$

and let

$$f_1 = f_1(N, q), f_2 = f_2(N, q), \frac{Q_0}{q} \leq f_1 < f_2 \leq \frac{N^{1/2}}{q}, \\ m_1 = \min\{f_1(N, N_j), f_1(N, N_{j+1})\}, m_2 = \max\{f_2(N, N_j), f_2(N, N_{j+1})\}.$$

Lemma 4.4. (*[5, Lemma 16.7.]*) *If the functions $f_1(N, q), f_2(N, q)$ are monotonic for q , then the following inequality holds*

$$\sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \leq \\ \leq \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2 m_1 \leq |K| \leq m_2} \int \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK. \quad (4.9)$$

Lemma 4.5. *For $\gamma \leq \frac{1}{8} - 4\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds*

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > N^{2\gamma+7\epsilon_0}}}^* \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.10)$$

Proof. It is sufficient to use Lemma 4.4 and the estimate (3.22) with (3.23). \square

Lemma 4.6. (*[5, Lemma 16.10.]*) For $\gamma \leq \frac{1}{8} - 5\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{2\gamma+7\epsilon_0} \\ q > N^{\gamma+5\epsilon_0}}}^* \int_{\frac{N^{3\gamma+12\epsilon_0}}{q} \leq |K| \leq \xi_1} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.11)$$

Lemma 4.7. (*[5, Lemma 16.13.]*) For $\gamma \leq \frac{1}{8} - 5\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\frac{\xi_1}{q} \leq |K| \leq \min\{\xi_1, \frac{N^{3\gamma+12\epsilon_0}}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.12)$$

Let ν be a positive real number, for example let $\nu \in [1, 2]$.

Lemma 4.8. (*[5, lemma 16.15.]*) For $\gamma \leq \frac{5(1+\nu)}{46+36\nu} - 6\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1^{1/(\nu+1)} \\ q > Q_0}}^* \int_{q^\nu \leq |K| \leq \frac{\xi_1}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.13)$$

It remains to estimate the integral over the domain 5.

Lemma 4.9. For $\gamma \leq \frac{1}{5+2\nu} - 6\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\frac{Q_0}{q} \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.14)$$

Proof. It follows immediately from the proof of Lemma 16.14 in [5] that for $\gamma \leq \frac{1}{6} - 5\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\frac{Q_0}{q} \leq |K| \leq Q_0} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.15)$$

Actually, we use Lemma 4.4 with

$$c_1 = Q_0, c_2 = \xi_1, f_1 = \frac{Q_0}{q}, f_2 = Q_0, m_1 = \frac{Q_0}{Q_1}, m_2 = Q_0.$$

It was proved in [5, (16.45)] that

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \frac{\overline{K}^{4\gamma+12\epsilon_0} Q^{6\gamma+1+20\epsilon_0}}{\overline{K} Q_1^2}. \quad (4.16)$$

Integrating over K and taking into account $m_2 \ll 1$ we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 Q^{6\gamma-1+22\epsilon_0}. \quad (4.17)$$

For the sum over j to be bounded by a constant, it is sufficient to have $\gamma \leq \frac{1}{6} - 5\epsilon_0$. Hence, it remains to prove that

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{Q_0 \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.18)$$

Using (2.8) and arguments similar to Lemma 4.4, we obtain

$$\begin{aligned} \frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{Q_0 \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK &\ll \frac{1}{TN} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \sum_{TQ_0 \leq |l| \leq T \min\{q^\nu, \frac{\xi_1}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right|^2 \\ &\ll \frac{1}{TN} \sum_{j: Q_0^{1-\epsilon_0} \leq N_j \leq \xi_1} \sum_{i: Q_0^{1-\epsilon_0} \leq N_i \leq \min\{N_{j+1}^\nu, \frac{\xi_1}{N_j}\}} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N\left(\frac{a}{q} + \frac{l}{TN}\right) \right|^2. \end{aligned} \quad (4.19)$$

Applying Lemma 3.1 with $Q_1 = N_j$, $Q = N_{j+1}$, $\kappa_1 = TN_i$, $\kappa = TN_{i+1}$ and taking into account that the number of summands in the sum over i is less than $c \log \log N_{j+1}$ we obtain

$$\begin{aligned} \frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{Q_0 \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK &\ll \\ &\ll \frac{|\Omega_N|^2}{N} \sum_{j: Q_0^{1-\epsilon_0} \leq N_j \leq \xi_1} \sum_{i: Q_0^{1-\epsilon_0} \leq N_i \leq \min\{N_{j+1}^\nu, \frac{\xi_1}{N_j}\}} (N_{j+1}^2 N_{i+1})^{2\gamma+7\epsilon_0} N_j^{\gamma-1+5\epsilon_0} \\ &\ll \frac{|\Omega_N|^2}{N} \sum_{j: Q_0^{1-\epsilon_0} \leq N_j \leq \xi_1} N_j^{(5+2\nu)\gamma-1+45\epsilon_0}. \end{aligned} \quad (4.20)$$

For the sum over j to be bounded by a constant, it is sufficient to have $\gamma \leq \frac{1}{5+2\nu} - 6\epsilon_0$. This completes the proof of the lemma. \square

Having set $\nu = \frac{3}{2}$, we obtain that for $\gamma \leq \frac{1}{8} - 6\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the first integral in the right side of (4.1) is less than $\frac{|\Omega_N|^2}{N}$. So the inequality (2.2) holds for $\gamma \leq \frac{1}{8} - 6\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ and Theorem 1.2 is proved.

As mentioned in the paper [6], the proof of Lemma 4.8 significantly uses the results of the paper [2]. The following version of Lemma 4.8 was proved in [6] by elementary methods with the use of the estimates of Kloosterman sums.

Lemma 4.10. (*[6, лемма 8.10.]*) For $\gamma \leq \frac{\nu-1/2}{10(1+\nu)} - 8\epsilon_0$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1^{1/(\nu+1)} \\ q > Q_0}}^* \int_{q^\nu \leq |K| \leq \frac{\xi_1}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (4.21)$$

Having set $\nu = \frac{3+\sqrt{34}}{2}$, we obtain that for $\gamma \leq \frac{1}{8+\sqrt{34}} - 8\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the first integral in the right side of (4.1) is less than $\frac{|\Omega_N|^2}{N}$. So the inequality (2.2) holds for $\gamma \leq \frac{1}{8+\sqrt{34}} - 8\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$. Hence, the following theorem is valid.

Theorem 4.1. *For any alphabet \mathcal{A} with*

$$\delta_{\mathcal{A}} > 1 - \frac{1}{8 + \sqrt{34}} = 0,9276 \quad (4.22)$$

the inequality (1.2) holds.

Remark 4.1. *It is proved [3] that $\delta_{10} = 0,9257\dots$. From this follows that the alphabet $\{1, 2, \dots, 10, 11\}$ seems to satisfy the condition of Theorem 4.1.*

Список литературы

- [1] J. BOURGAIN, A. KONTOROVICH. On Zaremba's conjecture, preprint available at arXiv:1107.3776(2011)
- [2] J. BOURGAIN, A. KONTOROVICH, P.SARNAK. Sector estimates for hyperbolic isometries. GAFA, 20(5):1175-1200,2010.
- [3] O. JENKINSON. On the density of Hausdorff dimensions of bounded type continued fraction sets: the Texan conjecture. Stochastics and Dynamics, 4 (2004), 63-76.
- [4] S.K. ZAREMBA. La méthode des "bons treillis" pour le calcul des intégrales multiples. In Applications of number theory to numerical analysis, pages 39-119. Academic Press, New York, 1972.
- [5] D.A. FROLENKOV, I.D. KAN. A reinforcement of the Bourgain-Kontorovich's theorem, preprint available at arXiv:1207.5168
- [6] D.A. FROLENKOV, I.D. KAN. A reinforcement of the Bourgain-Kontorovich's theorem by elementary methods, preprint available at arXiv:1207.4546
- [7] S.V. KONYAGIN. Estimates for trigonometric sums over subgroups and for Gauss sums. (Russian) IV International Conference "Modern Problems of Number Theory and its Applications": Current Problems, Part III (Russian) (Tula, 2001), 86–114, Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., Moscow, 2002.
- [8] N.M. KOROBOV. Exponential Sums and Their Applications. Kluwer Academic Pub,1992.